Interface dynamics and domain growth in thermally bistable fluids

Nir J. Shaviv and Oded Regev
Physics Department, Technion—Israel Institute of Technology, Haifa 32000, Israel
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Domain growth kinetics in a thermally bistable fluid with heat diffusion is studied. The time evolution of interfaces between the stable phases is calculated numerically in two dimensions and compared to some general results derived analytically. The qualitative behavior is found to be similar to the previously studied cases where fluid dynamics was neglected. There are, however, several important differences such as the value of the dynamical exponent, which determines the power law of the system's correlation length growth. The introduction of fluid motion into the model introduces additional properties, unfamiliar to previously studied systems, like the change of the pressure or the size of the system. This behavior is due to the advection of mass. The present model may have general relevance to any system modeled by a real Ginzburg-Landau-type equation coupled to fluid dynamical conservation equations. In particular, it is a step forward on the way to a faithful modeling of thermally bistable cloudy astrophysical media.

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I. INTRODUCTION

Thermally unstable fluid media appear quite often in nature. Optically thin plasmas subject to radiative cooling, which are thermally bistable, have been invoked in a number of astrophysical contexts (see, e.g., [1–5] and references therein) and in laboratory plasmas as well (e.g., [6]).

These systems are characterized by a heat equation, which for isobaric conditions can sometimes be casted in a form of a reaction-diffusion equation, resembling the usual real Landau-Ginzburg equation with a bistable potential functional (see Ref. [15]). However, since the medium is a fluid and the different stable phases have different temperatures (and thus densities), mass advection and not only heat diffusion is expected and fluid motion has to be included in the model.

Linear analysis methods exploring the existence of thermal instabilities and their properties have been extensively employed starting from the work of Field [7] (e.g., Refs. [8–13]). The problem has also been approached numerically in various astrophysical contexts [3,4,14]. An alternative approach to the nonlinear problem, considering the medium as a pattern forming system has been introduced by Elphick, Regev, and Spiegel [15]. They used the simplest approach to the dynamics of fronts separating the two stable phases, looking at a one-dimensional system, with the fluid dynamics suppressed. Following Zel'dovich and Pikel'ner [16], who recognized the importance of and defined a stationary (in one dimension) front existing for a constant, critical value of the pressure, the work in Ref. [15] described the structure and the dynamics of such a phase front and the interaction between different fronts. They found that on the way to a slow phase separation it is possible to trigger the formation of complex chaotic structures by small spatial perturbations in the cooling function.

This approach can be expanded to the full one-dimensional (1D) hydrodynamic model using a Lagrangian coordinate, as was shown by Meerson [17]. Indeed, the analysis of Elphick, Regev, and Shaviv [18] has shown that this case is completely equivalent to the purely thermal model. Aranson, Meerson, and Sasorov [19] have recently looked in detail into the 1D hydrodynamic problem, examining the effects of boundary conditions on the pressure development in the system. The analysis in Ref. [18] was done for the case of "open boundaries" (Neumann boundary conditions) and for a pressure equal to its critical value (when the potential functional is symmetric). With these boundary conditions the asymmetric case gives rise to a fast disappearance of the metastable phase and uniformity of the system. Reference [19] examined the case of "closed boundaries" with no mass flux allowed (Cauchy boundary conditions) and proved that the pressure will approach in time its critical value (symmetric potential functional) giving rise finally to a rather simple but very persistent pattern of the cold and hot phases coexisting for very long times. While the case studied in Ref. [19] is more appropriate to laboratory plasmas, astrophysical plasmas are more naturally modeled using free boundaries. The critical pressure may, however, be attained by some sort of confinement (e.g., gravitational), as suggested in Ref. [19], but on a large scale. The model in Ref. [18] can then be viable as a description of a smaller scale subsystem, after the pressure has become critical.

The extension of the problem to two dimensions is rather problematic, as the semianalytical methods used in Refs. [15] and [18] are not easily applicable. There seems to be no choice but to have recourse to heuristic analytical estimates or fully numerical modeling. Some observations, however, can be made at the outset. First, it is clear that in the purely thermal and isobaric case in more than one dimension a closed front is not stationary, even if the pressure is critical (see Ref. [15] for a proof). The fronts develop towards curve shortening [20] and their motion depends on the curvature. If the pressure is not critical, front motion is additionally driven by the asymmetry of the phases [21,22]. If many fronts are present and are close to each other, front interactions may also play a role. Aharonson, Regev, and Shaviv [23], have recently investigated numerically the behavior of thermally bistable fronts in two dimensions, without the inclusion of fluid motion and for uniform critical pressure systems. Their investigation showed that indeed the two-dimensional systems may exhibit a richer behavior than the 1D case when spatial perturbations are included. The complex nature of the cloud patterns was described with the help of the fractal dimension of the interfaces and the dynamical exponent (the exponent with which the correlation length grows as a function of time). Like the 1D models, this system can under certain spatial and temporal perturbations remain mixed and complex.

The aim of the present work, as a next step, is to extend the investigation of the 2D model by including fluid dynamical effects to the already existing physical effects of thermal diffusion and bistability. We shall also drop the assumption of a constant pressure, although, when the dynamic time is much shorter than the thermal one the isobaric assumption is excellent. The paper is organized in the following way. Section II describes the theoretical model and the known analytical and qualitatively expected results, some of them derived using statistical methods. In Sec. III the numerical computations which support the analytical estimates are described. We conclude with a discussion of the results, their significance and some of the consequences in Sec. IV.

II. THE THEORETICAL MODEL

A. Assumptions and equations

Consider a medium with a uniform chemical composition, characterized by the temperature $T(\mathbf{r},t)$, the pressure $p(\mathbf{r},t)$, and the velocity $\mathbf{v}(\mathbf{r},t)$ fields. We assume that the total local cooling function is given and that there is spatial thermal coupling via heat diffusion. The conservation of mass, momentum, and energy give the following set of equations (in addition, we include the perfect gas equation of state).

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 , \tag{2.1}$$

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right]+\nabla p=0 , \tag{2.2}$$

$$\frac{1}{\gamma-1}\left[\frac{\partial p}{\partial t}+\mathbf{v}\cdot\nabla p\right]+\frac{\gamma}{\gamma-1}p\nabla\cdot\mathbf{v}+\rho\mathcal{L}(T,p)-\nabla\cdot(\kappa\nabla T)=0 , \tag{2.3}$$

$$p-\frac{\mathcal{R}}{\mu}\rho T=0 . \tag{2.4}$$

The first equation implies conservation of mass which will be the cause for the advection, even with a uniform pressure. The heat equation includes both the cooling function, $\mathcal{L}(T,p)$ (see Refs. [5,15]), which is responsible for the bistability, and thermal diffusion. The heat diffusivity is assumed from here on to be of the form $\kappa=\kappa_0 T^{\alpha}$, with α constant.

We can relate this full set of equations to earlier works which used a single thermal equation (Refs. [15,23]) by assuming that (1) the pressure is constant; (2) the velocities are negligible.

The first assumption is reasonable if the dynamical time (defined as the sound crossing time) is much smaller than other relevant time scales (the cooling time and the thermal diffusion time). This approximation is generally very good in the astrophysical cases of interest (see Ref. [18]) and is referred to as the "short wavelength limit" (see Ref. [19]). In such a case, the pressure equilibrates to a constant value, imposed on the boundaries, before any significant thermal changes can occur, either by radiative cooling or by heat transport (isobaric conditions). The assumption is thus valid in the short wavelength limit when a fixed pressure is imposed on the boundaries. The second assumption is never strictly correct if there are density variations in the system giving rise to fluid motion because of mass conservation. However, in the 1D case a single thermal equation (sometimes called a "reduced equation," see Ref. [19]) is still available by introducing a Lagrangian mass coordinate of the spatial coordinate (Refs. 17,18]). In the 2D case a reduced equation with mass motion may still be viable if the motions in the plane are suppressed by some mechanism (e.g., magnetic fields). In such a case, we may have a slab in the x-y plane with fluid motions only in the z direction.

The first assumption reduces the above set of equations for the 1D case into a single equation:

$$\frac{\partial Z}{\partial t}=F(Z,p)+Z^{\lambda}\frac{\partial^2 Z}{\partial m^2} , \tag{2.5}$$

with the following definitions: $Z(\mathbf{r},t)\equiv T^{\alpha+1}$ and $F(Z,p)\equiv -\alpha T^{\lambda\alpha}\mathcal{L}(T,p)$. The variable m is a Lagrangian space variable and $\lambda=1-1/\alpha$. Equation (2.5) is in a nondimensional form. See Ref. [18] for details.

We have assumed in Refs. [15], [18], and [23], for the sake of convenience, that the cooling function can be approximated by a simple form, giving bistability and a possibility of a reasonable fit with the actual radiative cooling function. The form appropriate for (2.5) is

$$F(Z,p)=Z^{\lambda}\left[(Z_0-Z)^3-\Delta^2(Z_0-Z)-\beta\ln\left[\frac{p}{p_c}\right]\right] . \tag{2.6}$$

The nature of the equations is such that the roots of the cooling function are steady uniform solutions of the partial differential equation (PDE). For a certain pressure region, in the isobaric case, we have three roots, two stable solutions and an unstable one. Each one of these roots is a "thermal phase," having a different temperature and density but the same pressure (see Refs. [17,15]). The exact form of the cooling function in the astrophysical cases is of course much more complicated (see Ref. [5] and references therein), but the simple model describes the general results of the universality class. The form of the pressure term in the cooling function guarantees that at the critical pressure ($p=p_c$) the two phases are energetically symmetric, therefore, there is no favorable phase. In systems where $p>p_c$, the denser phase is favorable and the systems develop in such a way as to increase that phase's extent. For $p<p_c$, the situation is the opposite.

However, in general, the pressure is not necessarily constant even in the short wavelength limit. In the case of a temporally variable pressure imposed on the boundaries, the system's pressure will follow in time the boundary value while remaining fairly uniform in space. If the system is within rigid boundaries, the pressure will remain rather uniform (if the short wavelength limit is applicable), however, its average value does not have to remain constant. As it was shown in Ref. [19], and as we shall see further on, the pressure in such systems will tend to approach the critical value.

The system of equations for the multidimensional hydrodynamic case cannot be treated in way similar to the ones employed in Refs. [15] and [18]. We shall solve it numerically and describe the results in the next section. In the following subsections we shall attempt to obtain general global results on the system using heuristic analytical and statistical arguments. In the next subsection we derive in a heuristic way the correlation length behavior for hydrodynamic systems. Then we shall find some additional properties of such systems not directly related to that derivation.

B. The eikonal equation and its consequences

Consider a reaction-diffusion equation in the following scaled form:

$$\epsilon\frac{\partial u}{\partial t}=f(u)+\epsilon^2\nabla^2 u \ . \tag{2.7}$$

This is a generic equation for the field u and let the form of the bistable function, $f(u)$, be $u(1-u)(u-a)$. Looking at points on the interface between the phases of such a classical Landau-Ginzburg model, it can be shown that the following equation, referred to (see Ref. [22]) as the "eikonal equation" exists:

$$N+\epsilon K=c \ . \tag{2.8}$$

This equation describes the velocity of a point on the interface (N) as a function of the interface's local curvature (K) with the help of constants from the original equation including ϵ, a scaling constant, and c, the constant velocity of an interface with zero curvature, caused by the asymmetry between the phases (because $a\neq\frac{1}{2}$).

We cannot case the general hydrodynamic multidimensional case into a reduced equation of the type of (2.7). However, it is possible to explore the multidimensional problem for a special case in which the fronts are spherically symmetric, since the definition of a Lagrangian coordinate is then possible similarly to 1D. The 1D case gives rise to Eq. (2.5), which for $\alpha=1$ has the generic form. Looking now at a spherical D-dimensional interface with radius r, we may define the Lagrangian coordinate, m, by $dm=c_0 r^{D-1}dr$, where c_0 is related to some mean density of the phases. Proceeding as in Ref. [22] and noting that the PDE will have a first derivative correction, related to the curvature (see also Ref. [15]) we conjecture that the eikonal equation for this case will have the following form:

$$N_m+K=c_m \ , \tag{2.9}$$

where the subscript m refers to velocities in m space and ϵ has been scaled out for convenience. The velocity c_m is the one a plane front would have in mass space ($dm=\rho dx$). Thus, we assume essentially that the eikonal equation for the purely diffusive general multidimensional case is similar in its form to the equation holding in a system with advection but in the spherically symmetric case. We stress again that in the last case all velocities have to be expressed in mass space. Consequently, the equation has meaning only for spherical fronts, because in a more complex situation the Lagrangian coordinate *cannot* be defined.

We have $dm=N_m dt$, defining N_m. Comparing this with the above mentioned definition of m we obtain $N_m dt=c_0 r^{D-1}dr$.

We distinguish now between two types of behavior according to the value of the pressure.

(i) For $p\to p_c$, we have $c_m\to 0$ (symmetric phases). The front moves only because of its curvature (see Refs. [15,20]). The eikonal equation has then the form $N_m=-K=-[(D-1)/r]$. Thus

$$-\frac{(D-1)}{r}dt=c_0 r^{D-1}dr \ . \tag{2.10}$$

Integration yields

$$r^{D+1}-r_0^{D+1}=-\frac{D^2-1}{c_0}t \ , \tag{2.11}$$

which means that, as expected, the spherical cloud will shrink because of curvature driven motion. Define now the typical length scale of the growing domain, L, by $L^{D+1}\equiv r_0^{D+1}-r^{D+1}$. Since the spherical cloud shrinks, the growing domain is the ambient phase, whose typical length scale can be represented by L. $L(t)$ grows with time in this curvature driven motion as

$$L(t)\propto t^{1/(D+1)} \ . \tag{2.12}$$

We get in 2D, $L(t)\propto t^{1/3}$, and in 3D, $L(t)\propto t^{1/4}$.

(ii) For $c_m\gg K$, which happens when p is far from p_c, we have $N_m=c_m$. This is the case of pressure driven motion. Now $c_m dt=c_0 r^{D-1}dr$, yielding

$$r^D=r_0^D=\frac{c_m D}{c_0}t \ . \tag{2.13}$$

The cloud will grow or decay according to the sign of c_m and defining again the typical length scale of the growing domain as $L^D\equiv|r^D-r_0^D|$ we get

$$L(t)\propto t^{1/D} \ . \tag{2.14}$$

Hence, in 2D $L(t)\propto t^{1/2}$ and in 3D, $L(t)\propto t^{1/3}$.

All the above power law relations, derived for a spherical cloud, having meaning only as long as the cloud exists, i.e., before its possible disappearance. We now *con-*

jecture that the power law behavior of $L(t) \propto t^{1/n}$, is the same for both L, as defined above for the system with a spherically symmetric front, and the *correlation length*, ξ, in a general cloudy system viewed as a disordered medium. ξ is related to the correlation function, $C(\mathbf{r},t)$:

$$C(\mathbf{r},t) \equiv \langle [T(\mathbf{x}+\mathbf{r},t)-\langle T\rangle][T(\mathbf{x},t)-\langle T\rangle] \rangle \;, \tag{2.15}$$

by

$$C(\mathbf{r},t) \propto e^{-r/\xi} \; . \tag{2.16}$$

The only rationale behind this conjecture is dimensional. L is the only relevant length scale in the system. In the spherical cloud it is related to the radius, while in a disordered system it is the correlation length. We shall see in Sec. III that the numerical simulations indeed confirm this assumption (up to the accuracy we can measure) as depicted in Fig. 1.

C. The ambient pressure and cloud extinction

In the case of experiments in the lab, we can usually choose the boundary conditions and parameters if it is necessary, for example, to fit the pressure to its critical value. In the astrophysical context, however, the boundary conditions are notoriously problematic and we have essentially no control of the parameters. It is, thus, relevant asking what the probability is of having in an observed cloud system a certain parameter value, in particular, e.g., critical pressure. Consider an ensemble of cloudy systems with a distribution of initial cloud sizes and in different ambient pressures. Will we have a good chance to observe systems close to the critical conditions, or will they be scarce? This question is particularly relevant to our previous works since it was usually assumed that since clouds for which $p = p_c$ live much longer and prevail, systems in which clouds are observed have to be close to the critical conditions. It was also shown by Ref. [19] that under certain conditions the pressure equilibrates to the critical value.

Examine first one system from the ensemble having a specific pressure. The correlation length behavior, as a function of time, depends on how close is the system to the critical point. As explained in the previous subsection, there are two limits to the behavior of the system's correlation length. Assume that the system is started from some random initial configuration, containing structures on small scales. We expect, thus, that curvature driven motion will be important as long as the small structures are present. This is so because curvature driving is in such a case more important than pressure driving. In the case of Eq. (2.9), this means that $K \gg c_m$. Consequently we may consider this regime as one with effectively critical pressure. Later on, when only large structures remain, the role of the pressure in driving the motion becomes dominant, Consequently,

(i) for small t we are in a curvature driven motion regime and the correlation length behaves like $L_1 = a_1 t^{1/(D+1)}$, where a_1 is a constant (see Fig. 2).

(ii) For large enough t we are effectively far from the critical pressure, and we have $L_2 = a_2 y t^{1/D}$ with $y \equiv \pm |p - p_c|^{1/D}$ since c_m is to first order $c_m \propto (p - p_c)$ and a_2 is a constant. The $+$ sign refers to the case $p > p_c$, while the $-$ is for the opposite case. We have thus for any fixed t, two symmetric branches for L as a function of y (see Fig. 3). We shall henceforth deal only with the positive y branch, remembering that an equivalent branch for $y < 0$ exists too.

Assume now that the correlation length is given by a dimensionless function $f(x)$ of a dimensionless variable x, containing time and y:

$$L = L_0 f(x) \; . \tag{2.17}$$

The temporal variation will enter only through x (i.e., $x \propto t$) if we assume that scaling exists. Under this assumption

$$f(x \gg 1) = x^{1/D} \text{ and } f(x \ll 1) = x^{1/(D+1)} \; . \tag{2.18}$$

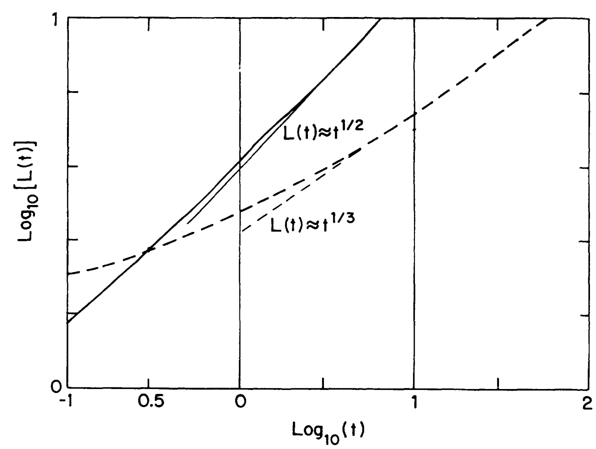

FIG. 1. Numerical result for the correlation growth in 2D systems. The correlation length grows as a function of time, giving different asymptotic power laws for the relation between L and t. The heavy lines are the numerical results while the thin ones are fits with the indicated power laws. Solid line corresponds to the case of pressure driven evolution and the dashed line is for curvature driven case.

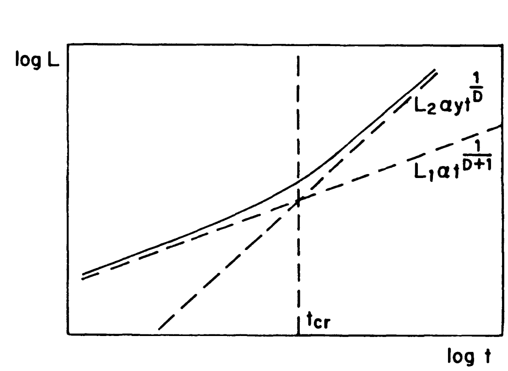

FIG. 2. Correlation length growth in hydrodynamic systems. Schematic drawing. For small t, the correlation length grows while the curvature is dominant, thus $L \propto t^{1/(D+1)}$. For large t the correlation length grows under the influence of the pressure, i.e., $L \propto t^{1/D}$. In between there is a crossover from one behavior to another. This happens at approximately $t = t_{cr}$, depending on the pressure.

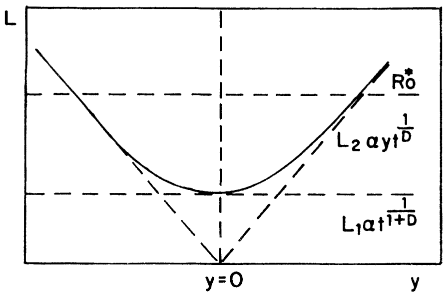

FIG. 3. Correlation length for systems with different pressures. Schematic drawing. $y \equiv \pm|p-p_c|^{1/D}$. For $y=0$ the correlation is always curvature dominated. At a distance of Δy (depending on the time), there is a change in behavior to pressure dominated growth. As y increases from this point, the correlation growth as a function of time is larger.

$x=1$ is the crossover point where $L_1=L_2$. The crossover time is

$$t_{\rm cr}=\left[\frac{a_1}{a_2 y}\right]^{D^2+D} \tag{2.19}$$

and since $x=1$ at $t=t_{\rm cr}$ we have

$$x=\left[\frac{a_2}{a_1}y\right]^{D^2+D} t \ . \tag{2.20}$$

Thus

$$L_0=a_2\left[\frac{a_1}{a_2}\right]^{D^2+D} y^{1-D-D^2} \ . \tag{2.21}$$

We next look at an ensemble of systems of clouds placed in different pressures. In Fig. 3 we can see the general behavior of the correlation length as a function of the boundary pressure. Around the center there is a region of width

$$\Delta y=2\frac{a_1}{a_2}t^{-[1/(D^2+D)]} \tag{2.22}$$

corresponding to pressure width

$$\Delta p=2\left[\frac{a_1}{a_2}\right]^{D} t^{-[D/(D^2+D)]} \ , \tag{2.23}$$

where the correlation length behaves as if the pressure is critical for any fixed t.

Consider now an initial distribution function of cloud sizes for a prescribed ambient pressure given by the function $g(R_0,p)$, defined in such a way that $g\,dR\,dp$ is the number of clouds in a given volume having initial sizes between R_0 and R_0+dR for a system in which the pressure is between p and $p+dp$. We write

$$g(R_0,p)=\frac{g_0}{R_0^*}\tilde{g}\left[\frac{R_0}{R_0^*}\right] \ , \tag{2.24}$$

with R_0^* being the typical initial size of a cloud, g_0 a suitable normalization constant, and $\tilde{g}$ a dimensionless function, whose p independence reflects the fact that we consider the same cloud size distributions for all pressures.

One can now calculate the number of clouds in the above mentioned central region of Fig. 3, where we have curvature influenced behavior. It is equal to the integral over the relevant pressure region (i.e., Δp) and over the sizes of clouds that still exist (those with initial sizes greater than the correlation length).

$$N_c=\Delta p\int_{L_1}^{\infty} g(L)dL \ , \tag{2.25}$$

with $L_1=a_1 t^{1/(D+1)}$ as given above in (i). This can be rewritten as

$$N_c=2a_0\left[\frac{a_1}{a_2}\right]^{D} t^{-[1/(D+1)]}h_1(\zeta) \ , \tag{2.26}$$

with $\zeta\equiv L_1/R_0^*$.

Like $\tilde{g}(u)$, $h_1(u)$ is a dimensionless function defined by

$$h_1(u)\equiv\int_u^{\infty}\tilde{g}(v)dv \ . \tag{2.27}$$

The number of clouds in the y region in which there is a pressure driven motion is

$$N_p=2\int_{\Delta p/2}^{\infty}\int_{L^2}^{\infty} g(L)dL\,dp \ , \tag{2.28}$$

where $\Delta p/2=(a_1/a_2)^D t^{-[1/(D+1)]}$ and $L_2=a_2 y t^{1/D}$ as given in (ii). This can be written also as

$$N_p=2g_0\int_{\Delta p/2}^{\infty} h_1\left[\frac{L_2}{R_0^*}\right]dp \ . \tag{2.29}$$

Remembering that $dp=Dy^{D+1}dy$ (for $y>0$) and defining another dimensionless function

$$h_2(u)\equiv\int_u^{\infty} dv^{D-1}h_1(v)dv \tag{2.30}$$

gives

$$N_p=2g_0\left[\frac{R_0^*}{a_2}\right]^{D}\frac{1}{t}h_2(\zeta) \ . \tag{2.31}$$

Therefore, the ratio between the number of clouds in the different regimes is

$$\frac{N_c}{N_p}=\zeta^D\frac{h_1(\zeta)}{h_2(\zeta)} \ . \tag{2.32}$$

Examine now this result for two typically expected initial cloud size distributions. If we start with an exponential distribution, having a typical cloud size of R_0^* initially, the ratio between the number of clouds in the curvature driven regime and in the pressure driven one is proportional to ζ. Therefore, most clouds will seem to develop according to pressure driven evolution (i.e., p effectively different from p_c) until the smallest correlation length in the ensemble of systems will reach the typical initial cloud size (i.e, $\zeta\approx 1$). The majority of clouds will seem then to develop according to curvature driven evolution (i.e., p effectively p_c).

In the case of an initial power law distribution, we have

$$g(R_0) \propto \frac{1}{R_0^{\Gamma}} \Longrightarrow \frac{N_c}{N_p} = \Gamma - (1+D) \ . \tag{2.33}$$

Thus, for a power law initial distribution, there will always be a constant ratio between the number of clouds in the curvature driven regime and the pressure driven one. It will be less than one (i.e., dominated by pressure driven clouds) only if we have $1+D < \Gamma < 2+D$, since because of obvious constraints, we must always have $\Gamma > 1+D$.

It is therefore not obvious that while examining an ensemble of cloudy systems at different pressures, we will indeed see, as time goes on, relatively more and more clouds in the curvature driven regime. In the case of an initial power law distribution of clouds, this assumption will be correct only if there are enough small clouds (Γ is large enough) to start with. When we have an initial distribution of clouds with a typical initial radius, such as an exponential distribution, the assumption will be viable if the correlation length in the curvature drive zone is larger than the typical initial cloud size, which of course happens always for large enough t.

D. Relation between fractal dimension, dynamical exponent, pressure evolution and the cloud complex size

The difference between the model described here and the usual reaction-diffusion type model [Eq. (2.7)] is caused, as mentioned before, by the fact that the two thermal phases have different densities and thus interface motion has to be accompanied by advection of mass. This is so in all but rather contrived cases, in which fluid motion is suppressed in certain directions. In a system with an average pressure different from the critical value, one of the phases is more stable and will tend to grow. Since the densities of the phases are different, the average pressure cannot remain constant in systems with constant volume and mass (rigid boundaries). In the case of free boundaries, where the system is embedded in a constant ambient pressure, the volume cannot remain constant, i.e., the cloud complex will either have to shrink or to expand.

Consider a cloudy system characterized by a correlation length L. A typical structure in the system, a "blob," should have a size of the order of L. As it was shown in II B the mass transfer rate in a spherical blob evolving by pressure driven interface motion($p \neq p_c$) is given by $dm_{\text{sph}} = c_m dt$. For a nonspherical typical blob the net change is $dm_{\text{blob}} = \beta c_m dt$, with β being some geometrical factor describing the average blob, which cannot be given by these heuristic arguments.

The total mass transfer rate between the phases in the entire system should, thus, be

$$dm = \frac{A}{A_0} \beta c_m dt \ , \tag{2.34}$$

where A / A_0 is the ratio between the total interface area of the growing phase, whose size is of the order of magnitude of the entire system, and the interface area of the aforementioned average blob.

However, in a complex system of size S^D (if scaling exists) in the area of objects the size of the system is related to the area of smaller objects (such as objects the size of the correlation length) via a relation containing the fractal dimension, α_f:

$$A = A_0 \left[\frac{S}{L} \right]^{\alpha_f} \ . \tag{2.35}$$

The meaning of α_f can be demonstrated by examining what happens to an object when the size of the system is rescaled. If the objects grows as the system is enlarged, then $\alpha_f = D-1$. If the object is made up of small bodies, whose number increases when the system expands, then $\alpha_f = D$. In the case of a real cloud complex, we have, of course, an intermediate dimension of $D-1 > \alpha_f > D$. The total mass transfer rate between the phases is thus given by

$$dm = \left[\frac{S}{L} \right]^{\alpha_f} \beta c_m dt \ . \tag{2.36}$$

Consider now a system in which the dilute phase, having a constant density, ρ_1, occupies a volume V_1 and a dense (ρ_2) phase's volume is V_2. Assuming that the masses of the phases are close to each other

$$m_1 = \rho_1 V_1 \simeq \rho_2 V_2 = m_2 \ . \tag{2.37}$$

This assumption is made in order to focus on systems, which are not about to become uniform in very short times (this happens when one phase's mass is significantly larger that the other's one). In addition, we assume that $\rho_1 \ll \rho_2$ as is the case in astrophysical applications. In this case, the volume of the dilute phase is close to the systems volume $V = V_1 + V_2$. We would like to examine what happens to the system's pressure as a function of time in two cases, differing from each other by the nature of the boundary conditions. In both cases, we assume that we start from pressures quite close to the critical pressure, p_c.

With the above assumptions we have

$$\rho_1 = \frac{m_1}{V} = \frac{m_1}{S^D} \rightarrow m_1 = \rho_1 S^D \ . \tag{2.38}$$

The pressure is determined by the density and temperature of the dilute phase (ρ_1 and T_1) via the perfect gas equation of state. As those change, the pressure can change too, and we have

$$\frac{dp}{p} = \frac{d\rho_1}{\rho_1} + \frac{dT_1}{T_1} \ . \tag{2.39}$$

Since we expect that significant global changes in the system caused by mass transfer between the phases occur on a time scale much longer than the cooling time we may assume that thermal balance prevails, i.e., $\mathcal{L}(\rho_1, T_1) = 0$. This gives a relation between ρ_1 and T_1. Other assumption may give rise to other relations, the important fact at this point being that (2.39) may be written as

$$\frac{dp}{p} = \frac{d\rho_1}{\rho_1} \left[1 - \frac{dT_1}{d\rho_1} \frac{\rho_1}{T_1} \right] \equiv \eta \frac{d\rho_1}{\rho_1} \ , \tag{2.40}$$

where the last equality defines η.

In the case of rigid boundaries, the system mass and volume remain constant, and the pressure can change. Using (2.40) and (2.38) and substituting the expression from (2.36) as dm_1 we obtain

$$\frac{dp}{p}=\beta S^{\alpha_f-D}L^{-\alpha_f}\frac{\eta c_m}{\rho_1}dt \; . \tag{2.41}$$

Close to the critical point, to first order $c_m=-c_0(p-p_c)$, where c_0 is a constant. Defining now $\tilde{p}\equiv p-p_c$, (2.41) can be written to first order in $|p-p_c|$:

$$\frac{d\tilde{p}}{p_c}=-\beta c_0 S^{\alpha_f-D}L^{-\alpha_f}\frac{\eta\tilde{p}}{\rho_1}dt \; . \tag{2.42}$$

If the correlation length grows like $L=L_0t^z$, we find

$$\frac{d\tilde{p}}{\tilde{p}}=Q_r t^{-z\alpha_f}dt \; , \tag{2.43}$$

where $Q_r\equiv-\beta c_0 S^{\alpha_f-D}L_0^{-\alpha_f}\eta/\rho_1$.

In the limit of p close to p_c we may consider Q_r as a constant and integrate Eq. (2.43) under this assumption to give the approximate result

$$\tilde{p}(t)=\tilde{p}_0\exp\left[-\left[\frac{t}{\tau}\right]^{1-z\alpha_f}\right] , \tag{2.44}$$

where $\tau\equiv[Q_r/(1-z\alpha_f)]1/(1-z\alpha_f)$.

This formula gives a relation between the pressure change, the dynamic exponent and the fractal dimensions of the clouds in closed systems (fixed mass and volume), when the pressure is close to the critical one, i.e., after a sufficient time.

In systems with free boundaries, placed in an ambient gas with a constant pressure, the whole cloud complex may grow or shrink. If the dilute phase is more (less) stable its volume will increase (decrease) in time. Thus, from (2.36) and using (2.38) (assuming that ρ_1 does not vary significantly) to give $dm_1/m_1=D\,dS/S$ we get

$$S^{D-\alpha_f-1}dS=Q_f t^{-z\alpha_f}dt \; , \tag{2.45}$$

where $Q_f\equiv(\beta c_m/D\rho_1)L_0^{-\alpha_f}$ and is assumed constant. This integrates to

$$S(t)^{D-\alpha_f}=S_0^{D-\alpha_f}+Q_f\frac{D-\alpha_f}{1-z\alpha_f}t^{1-z\alpha_f} \; . \tag{2.46}$$

This formula gives the relation between the dynamical exponent, the fractal dimension and the system's size in the case of open boundaries and fixed pressure.

The two cases, as expressed by equations (2.44) and (2.46), describe typical features of systems in which the density of the two phases is different and fluid flow is necessary to ensure mass conservation. In reaction-diffusion systems without the constraint of a conservation of a physical quantity, internal changes of the system influence neither the size of the system nor the "external field" (like the magnetic field in spin systems). The first case of closed boundaries is better fit for numerical simulations, however, in an astrophysical system we expect that the open boundary case is more realistic, at least on smaller scales.

III. TWO-DIMENSIONAL NUMERICAL SIMULATIONS

Numerical simulations were performed to assess the validity of the analytical estimates and to better understand the 2D behavior, as compared to the completely solvable 1D model. The simulations were performed on a 200×200 square grid with a spacing of one length unit (the Field length), using a well known finite difference scheme: The alternating direction implicit (ADI) splitting method (see, e.g., Ref. [24]) for the solution of the heat equation. The continuity and momentum equations were solved using a standard explicit scheme, coupled to the ADI scheme (which is implicit).

The initial conditions for most simulations were of a random nature reflecting excitation on small scale (regions with size of between 3×3 and 10×10 grid cells were assigned to have different densities, equal to one of the stable phase densities). The ratio between the number of cells of lower density and cells of high density was usually inversely proportional to the density ratio, giving equal masses for both phases. Test runs were conducted first with initial conditions appropriate to one spherical cloud in the central region of the system.

Two types of boundary conditions were used. The first, referred to as "free" boundary conditions is

$$\frac{\partial \mathbf{v}}{\partial n}=0 \; , \quad \frac{\partial T}{\partial n}=0 \; , \quad p=p_0 \tag{3.1}$$

on the boundary, where n denotes the normal direction.

The second type, referred to as "rigid walls" boundary condition is, using the same notation:

$$v_n=0 \; , \quad \frac{\partial T}{\partial n}=0 \; , \quad \frac{\partial p}{\partial n}=0 \tag{3.2}$$

on the boundary.

Conditions of type (3.2) are not natural when one has in mind an astrophysical system but they give rise to some interesting features brought about by the global mass conservation and might apply on a large scale due to magnetic or gravitational confinement (Ref. [19]). The free boundary conditions are appropriate for a system with no defined boundaries, constrained by an ambient pressure of the surroundings. Here no mass conservation is imposed and the system's volume may change.

As a test we have computed with the help of our code the evolution of a single circular "cloud" of radius R_0. The evolution is quantitatively different from that of the purely thermal diffusive (no hydrodynamics) model (like the one in Ref. [23]). For a pressure equal to its critical value the evolution is driven by the curvature. We expect, using (2.11) that

$$R_0^3 - R^3 \propto t \ . \tag{3.3}$$

If the pressure is different than critical, using (2.13), we expect that

$$R_0^2 - R^2 \propto t \ . \tag{3.4}$$

The simulations, performed with free boundaries, gave excellent agreement with (3.3) and (3.4). In Fig. 4, the velocity field during the circular cloud evaporation is shown. In the same problem, but with rigid boundaries, the conserved total mass does not allow the cloud to evaporate. This happens even if the initial pressure is different from critical, since during the evolution of the cloud, the pressure stabilizes on a value that would stop the curvature driven curve shortening. The ultimate pattern will arise when the pressure is such as to offset the curve shortening tendency and the pattern is *stable.* A straight single front with critical pressure is such a possibility.

We have next tested the conjecture that domain growth in a randomly initiated system will proceed according to the same dynamical exponent as in the spherical case. Starting the system from random initial conditions and keeping the external pressure equal to its critical value we get first a short transient, during which the fronts themselves acquire their stable internal structure. Next the small structures start merging, increasing slowly the correlation length, or the "average" size of a cloud (or intercloud region). The evolution of such a cloud complex can be seen in a series of snapshots in Fig. 5.

Computing the correlation length from the correlation function as in (2.15), a good fit to $L \propto t^{1/3}$ is obtained, as predicted by the heuristic arguments of Sec. II B.

Changing the ambient pressure from the critical value, introduces growth driven by the asymmetry of the phases. Again, a good fit to the prediction that correlation length grows with time as $L \propto t^{1/2}$ for two-dimensional system is found after a sufficiently long transient. Note that for the purely thermal diffusive case, the domain growth of a nonconserved order parameter would be $L \propto t$ (Refs. [23,25]). In Fig. 1 the numerical results for the above mentioned two cases are depicted.

When rigid wall boundary conditions are applied the total mass is conserved. In Sec. II D (see also Ref. [19]) it was predicted that under these circumstances the pressure will approach its critical value. In Fig. 6 we depict the change in the pressure in time in the numerical simulation. The initial pressure is smaller than the critical one, but as time passes the pressure approaches its critical value. Thus, although $p \neq p_c$ as $t \to \infty$, the pressure

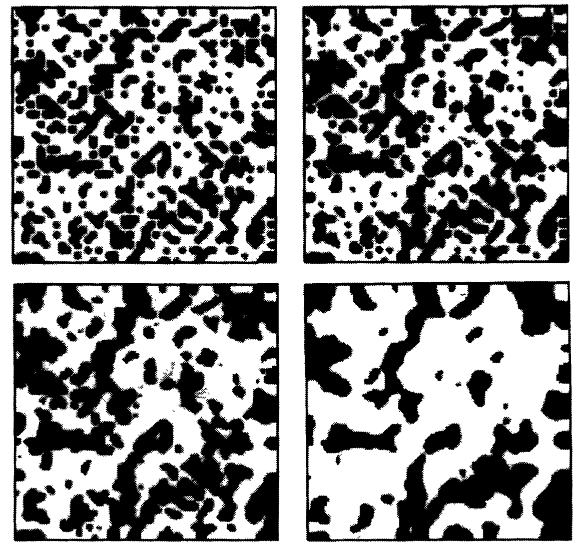

FIG. 5. A series of pictures describing the evolution of a cloud complex. The correlation length grows in the four pictures, slowly erasing the smaller details. In this series, $p = p_c$ and therefore $L \propto t^{1/3}$. The upper left picture is chronologically the first and the lower right the last.

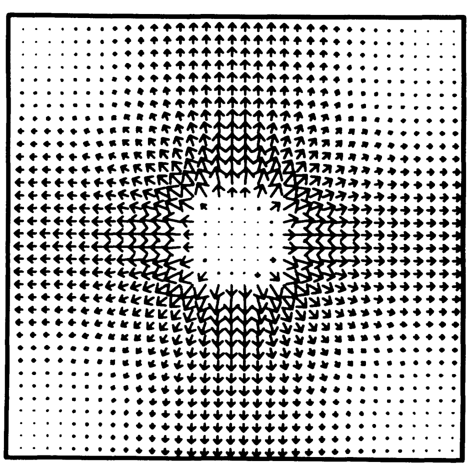

FIG. 4. Evaporation of a 2D cloud. Around the rim of the cloud we see large velocities of the evaporated cloud. Because of conservation of mass, these velocities must decrease with the distance from the center of the cloud.

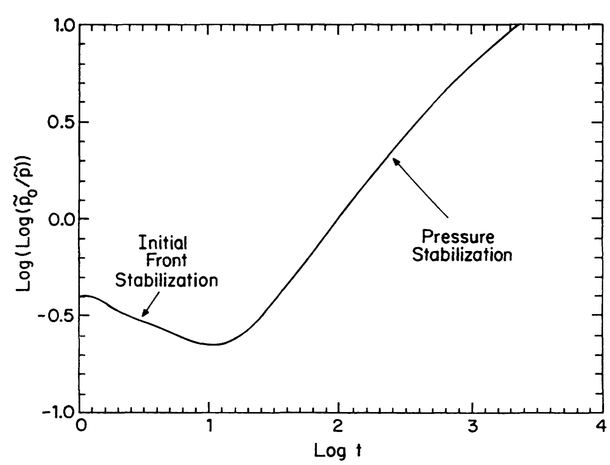

FIG. 6. Pressure equilibration in a bounded system. As predicted in [9] and shown here in (2.44), the pressure approaches the critical value in systems with rigid boundaries (imposing mass conservation). The log in both axes is to the base 10.

gets close enough to p_c (relatively to the curvature effect) for L to be $L \propto t^{1/3}$. In the analytical results, we have seen that there exists a relation between the pressure change, the fractal dimension, and the dynamical exponent, $z \equiv 1/n$. The asymptotic fit to formula (2.44) is good, but the accuracy with which α_f can be measured is unfortunately very poor for the present resolution. It seems that the resolution has to be increased by an order of magnitude in order to be able to infer a reliable result on α_f. This requires a substantial increase in computing resources (the runs reported here take a few hours CPU on a RISC computer workstation).

IV. SUMMARY

We conclude by briefly summarizing the main findings of this paper. A model of a fluid medium with heat transport and a nonlinear bistable heat source term was investigated here analytically and numerically. This work is a continuation of a series of investigations (Refs. [15], [18], and [23]). Here, in addition to those previous works, the effects of fluid advection are included in the multidimensional case.

Qualitatively, the picture described in our first work [15] still holds. The two phase medium develops in time with small structures gradually merging and forming larger uniform regions. The complex boundaries become smoother and shorter. This evolution may be affected by spatiotemporal persistent perturbations or by confinement of the system in a closed "box." In the former case complex pattern may persist for very long times like in the purely thermal case [23]. Mass conservation in the latter case does not allow uniformization of the whole system and a pattern with a simple structure the size of the whole system persists (Ref. [19]).

Quantitatively, we find here that the dynamical exponents governing the growth of the correlation length in unforced systems differ from the one found for the purely thermal-diffusive case. The analytical and statistical estimates, based on heuristic arguments, predict the correlation length growth as a function of time and numerical simulations support these predictions for the two-dimensional case.

TABLE I. Asymptotic behavior of correlation length for different cases.

System			Results	
Dimension	Pressure	v	$L(t) \propto$	Remarks
1	$=p_c$	$\neq 0$	$\log(t)$	front interaction
1	$\neq p_c$	$\neq 0$	t	
D	$=p_c$	$=0$	$t^{1/2}$	diffusion only
D	$\neq p_c$	$=0$	t	diffusion only
D	$=p_c$	$\neq 0$	$t^{1/(D+1)}$	curvature driven
D	$\neq p_c$	$\neq 0$	$t^{1/D}$	pressure driven

The various cases are summarized in Table. I.

These values of the dynamical exponents fit well the 2D numerical results only for sufficiently large times. Even when $p \neq p_c$, for small enough L the curvature can be high enough as to be the major factor in the evolution.

In the case of rigid wall boundaries, we have found a way to show (in a different way from Ref. [19]), and verified it numerically, that the pressure as $t \rightarrow \infty$ goes to the critical pressure, where again the curvature plays the main role.

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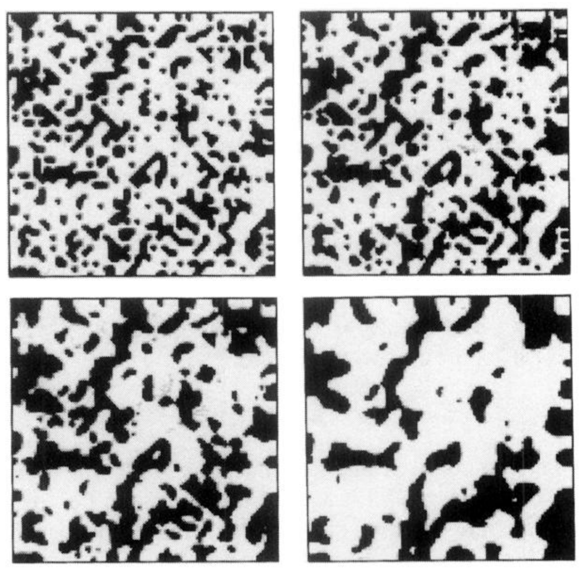

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